Spectral Analysis

- A Fourier series expansion decomposes a periodic function into an infinite sum of sin and cos functions.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi n x}{P}\right) + b_n \sin\left(\frac{2\pi n x}{P}\right)$$

- What do the coefficients, a_n and b_n mean?

Inner Products

- The procedure to obtain the coefficients is known as an inner product

$$a_n = \frac{2}{P}\int_{x_0}^{x_0+P} f(x) \cdot \cos\left(\frac{2\pi n x}{P}\right) dx\,, \qquad b_n = \frac{2}{P}\int_{x_0}^{x_0+P} f(x) \cdot \sin\left(\frac{2\pi n x}{P}\right) dx$$

- The inner product is a generalization of the dot product
- A dot product applies only to vector spaces, an inner product is any operation which behaves like a dot product, but is valid in any space

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_1\hat{x}_1 + a_2\hat{x}_2 \ldots) \cdot (b_1\hat{x}_1 + b_2\hat{x}_2 \ldots) = a_1 b_1 + a_2 b_2 + \cdots$$

- The relevant space here is function space. Imagine a function as a vector where the dimensions are the points on the x axis and the coefficients are the function values $f(x)$

- This leads to the definition of the inner product in a function space

$$\langle f(x), g(x) \rangle = \int_a^b f(x)g(x)\,dx$$

- By calculating the inner product we answer the question "How much do of function A is in function B?" This is what the coefficients of the Fourier Series mean
- We can define these more efficiently by replacing sin and cos using Eulers law

$$c_n = \frac{1}{P}\int_{x_0}^{x_0+P} f(x) \cdot e^{-i\frac{2\pi n x}{P}} dx$$

- This is potentially valuable information if we are analyzing a system that is fundamentally tied to oscillations, such as sound, electronics, telegraphy.
- It is also useful in problems that we might not initially associate with periodicity, like imaging, compression and solving differential equations

Exercise

- Write a function which computes the Fourier series (up to the nth term) of a function over a defined interval.

- Use this definition: $c_n = \frac{1}{P}\int_{x_0}^{x_0+P} f(x) \cdot e^{-i\frac{2\pi nx}{P}} dx$

Exercise

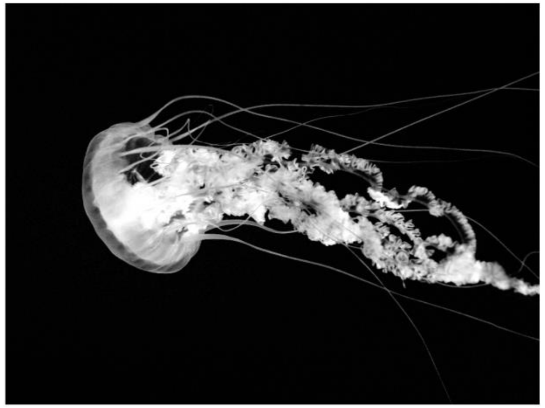
- The Fourier transform is like an extreme Fourier series (we'll talk about it soon). But it essentially takes a function an decomposes it into all of its component frequencies.
- Make a matlab function which plots a sin wave with a given frequency
- Then, use the built in matlab function: `fft()` To compute the fourier transform of that sine wave. Plot both the original curve and the *absolute value* of fourier transform.

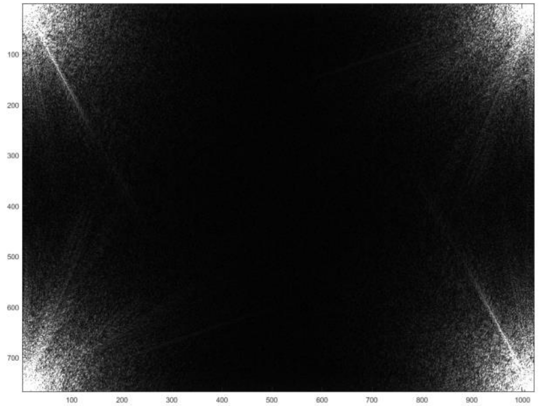
- What is the output of FFT? What are we plotting? Why do we take the absolute value? What should be on the x-axis?

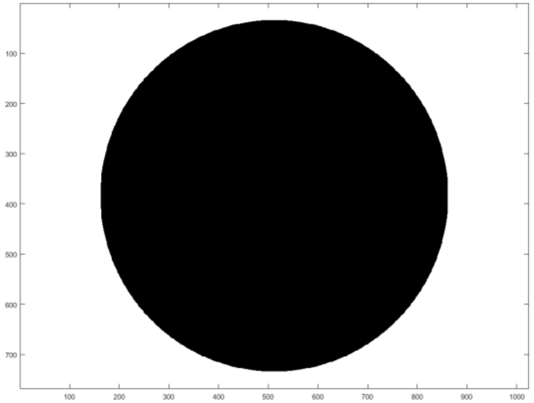
Applied FFT: Image processing

- We can use the FFT to compress images

Image → FFT2(image) .* Low_pass → IFFT2







- Original image was 1024x728x32bit ≈ 3.145 MB
- Image on right equivalent to ≈ 0.4 MB!!

Applied FFT: Image processing

Write your own image compressor. This image compressor should

1) Load in a greyscale BMP image
2) Use FFT2 to generate the 2D transform of the image
3) Apply a filter of your choosing
4) Save the filtered data to a file (in some format you chose… *csvwrite()* might be a good option here)
5) Reconstruct the image using IFFT2 on the filtered data.

Verify that the reconstructed image looks passably similar to the original bitmap. Check the difference in file-size!